Fixed-point tile sets and their applications

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joint work with

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Motivations for the definition: a simple framework for natural problems from logic, dynamical systems,... even from physics.

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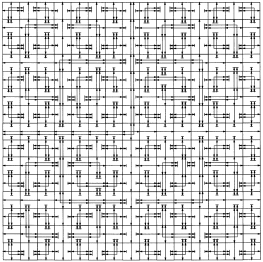
Theorem (Berger): There exists a tile set that allows some tilings but only aperiodic tilings are possible.



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- Tilings aperiodic, but close to periodic;
- There are periodic configurations that are almost tilings (sparse set of tiling errors)

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Such configurations do exist. Moreover, they can be enforced by tiling rules!

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Really simple?
Not in terms of the number of tiles.

Fix a tile set τ and number $N > 1$.

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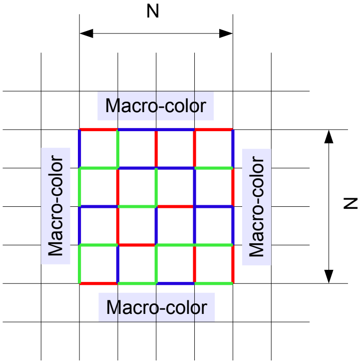
Some set ρ of $N \times N$-macrotiles is *simulated* by τ if every τ-tiling can be uniquely split into macrotiles by $N \times N$ grid.

Macro-tile:



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Example 2: a tile set that simulates a trivial tile set

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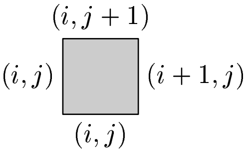

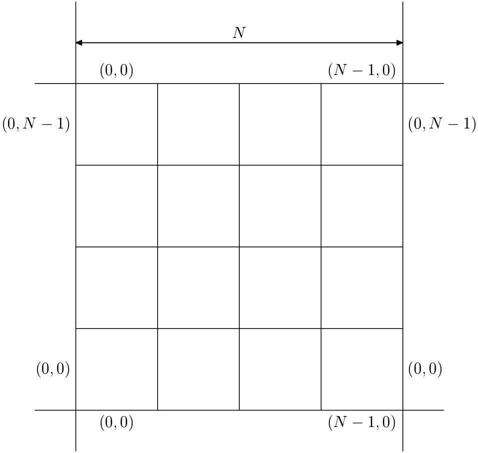
N
(0, 0)
(N − 1, 0)
(0, N − 1)
(0, N − 1)
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Folklore: many of aperiodic tile sets are self-similar.

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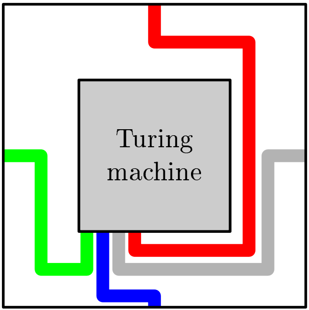
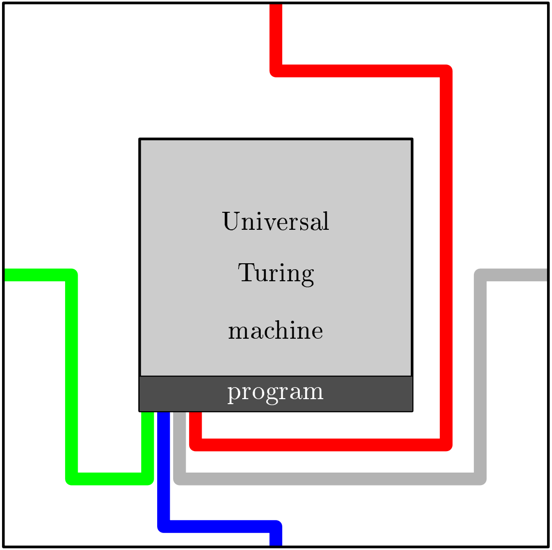
- set of tiles (a subset of C^4) presented as a predicate $R(x_1, x_2, x_3, x_4)$ whose arguments are bit strings
- tile set is presented as TM that accepts quadruples of colors that are tiles

Implementation scheme:



Using $O(1)$ additional colors: universal TM + program



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Technical remark: easy to implement a UTM that can access the bits of the simulating program.

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checking the program

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- strong aperiodicity (not hard: some detail below)

What else can we achieve with the *fixed-point* construction of aperiodic tilings?

- variable zoom factor (easy)
- undecidability (easy)
- robustness (tricky: some detail below)
- strong aperiodicity (not hard: some detail below)
- high Kolmogorov complexity (more involved, beyond this talk)

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B_ε: Bernoulli distribution where each cell belongs to a random set with probability ε and different cells are independent

The notion of "sparse set" is reasonable if for small enough ε a B_ε-random set is sparse with probability 1

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Distance in a square: the fraction of cells where two configurations differ.

Theorem: there exists a tile set τ such that for small enough ε the following is true for B_ε-almost all sets H:

every tiling of $\mathbb{Z}^2 \setminus H$ is at least $1/10$-Besicovitch far from every periodic mapping

Making the construction robust

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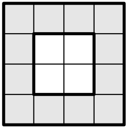
1. introduce redundancy (every tile “knows” information about its neighbors) $\Rightarrow$ we correct small errors (e.g., 2×2 holes)



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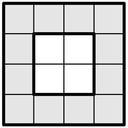

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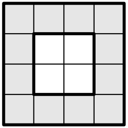

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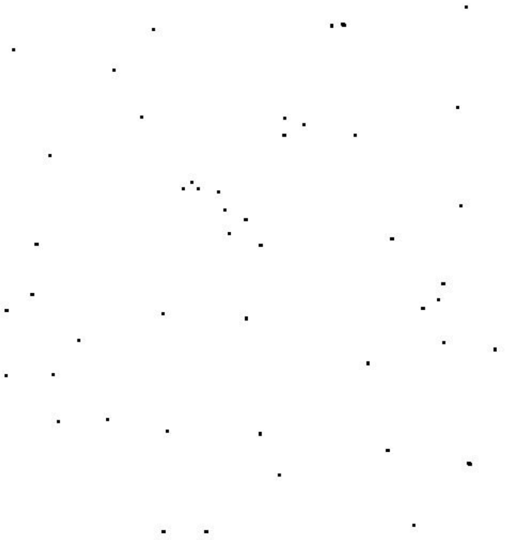
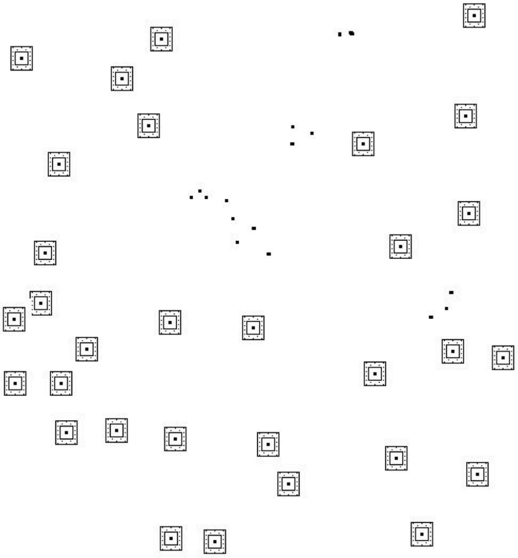
3. a real miracle: we can correct a *random* set of miracles (with prob 1)

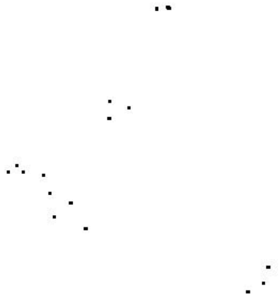
A B_ε-random set consists of isolated "islands" of different levels



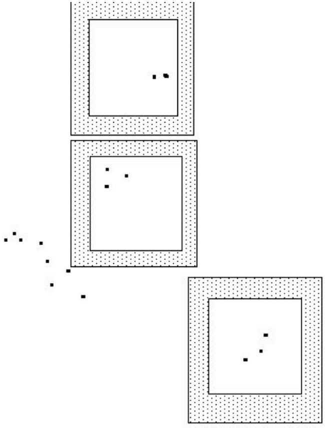
Isolated 0-level islands:



Clean up 0-level islands:

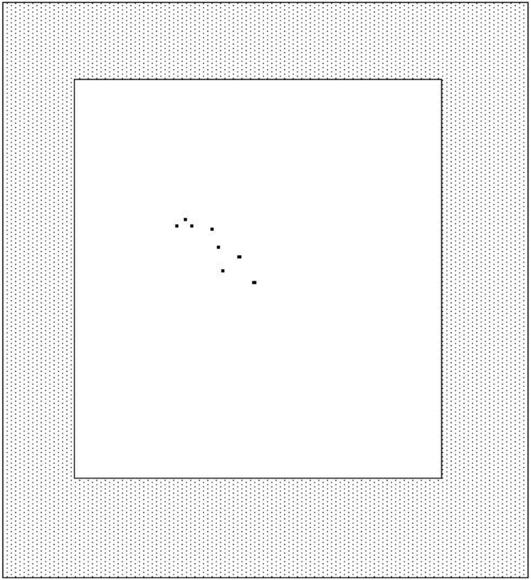


Isolated 1-level islands:



Clean up 1-level islands:

2-level island:



With probability 1 the cleaning procedure converges. Moreover, with probability 1 only the fraction $O(\varepsilon)$ of points is involved in the procedure.

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Implement the Thue–Morse substitution rule:

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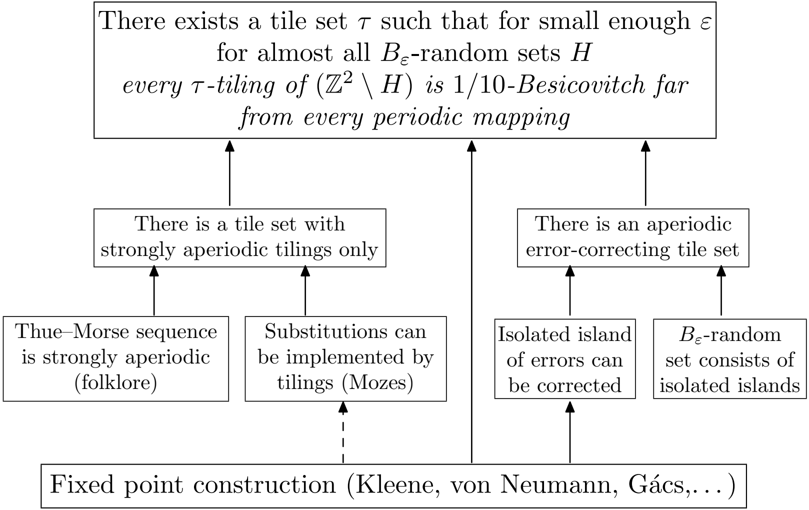
Lemma. The limit configuration of the TM substitution rule is strongly aperiodic.


There exists a tile set τ such that for small enough ε for almost all B_ε-random sets H *every τ-tiling of $(\mathbb{Z}^2 \setminus H)$ is 1/10-Besicovitch far from every periodic mapping*
There is a tile set with strongly aperiodic tilings only
There is an aperiodic error-correcting tile set
Thue–Morse sequence is strongly aperiodic (folklore)
Substitutions can be implemented by tilings (Mozes)
Isolated island of errors can be corrected
B_ε-random set consists of isolated islands
Fixed point construction (Kleene, von Neumann, Gács,. . .)

Thank you!