

Circular Proofs for Gödel-Löb Logic

Daniyar S. Shamkanov
daniyar.shamkanov@gmail.com

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Axioms

- ▶ Boolean tautologies
- ▶ $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- ▶ $\Box(\Box A \rightarrow A) \rightarrow \Box A$ (Löb's axiom)

Rules

$$\frac{A, A \rightarrow B}{B} \qquad \frac{A}{\Box A}$$

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GL is sound and complete w.r.t. the arithmetical semantics, where $\Box A$ corresponds to "A is provable in Peano arithmetic".

Sequent calculus syntax

Formulas are given by:

$$A ::= P \mid \overline{P} \mid \top \mid \perp \mid (A \wedge A) \mid (A \vee A) \mid \Box A \mid \Diamond A.$$

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Sequents are finite multisets of formulas.

For a sequent $\Gamma = A_1, \dots, A_n$, its intended interpretation as a formulas is

$$\Gamma^\# := \begin{cases} \perp & \text{if } n = 0, \\ A_1 \vee \dots \vee A_n & \text{otherwise.} \end{cases}$$

Standard sequent calculus $K4_{Seq}$

Axioms

$$\Gamma, A, \bar{A} \quad \Gamma, \top$$

Rules

$$\wedge \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \quad \vee \frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\Box \frac{\Gamma, \Diamond \Gamma, A}{\Diamond \Gamma, \Box A, \Delta}$$

Standard sequent calculus GL_{Seq}

$GL_{Seq} = K4_{Seq} + \text{the modal rule } \Box_{GL}$

$$\Box_{GL} \frac{\Gamma, \Diamond \Gamma, \Diamond \bar{A}, A}{\Diamond \Gamma, \Box A, \Delta}$$

Circular sequent calculus GL_{CSeq}

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A circular proof of the Löb's axiom

$$\begin{array}{c}
 \boxed{\square} \frac{\square P \wedge \bar{P}, \Diamond(\square P \wedge \bar{P}), P}{\square P, \Diamond(\square P \wedge \bar{P}), P} \\
 \wedge \frac{\quad \bar{P}, \Diamond(\square P \wedge \bar{P}), P}{\square P \wedge \bar{P}, \Diamond(\square P \wedge \bar{P}), P} \\
 \vee \frac{\Diamond(\square P \wedge \bar{P}), \square P}{\Diamond(\square P \wedge \bar{P}) \vee \square P}
 \end{array}$$

Theorem

$$GL_{Seq} \vdash \Gamma \Leftrightarrow GL_{CSeq} \vdash \Gamma$$

An application: Lyndon interpolation syntactically

Lyndon interpolation

Craig interpolation

If $L \vdash A \rightarrow B$, then there is a formula C containing only common variables of A and B such that $L \vdash A \rightarrow C$, $L \vdash C \rightarrow B$.

Lyndon interpolation

is a strengthening of the Craig one by the additional requirement:

every propositional variable that has a positive (negative) occurrence in C must also have positive (negative) occurrences both in A and B .

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Interpolation properties of GL

- Craig (Smoryński 1978, Boolos 1979)

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- ▶ Uniform (Shavrukov 1993)

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Interpolation properties of GL

- ▶ Craig (Smoryński 1978, Boolos 1979)
- ▶ Uniform (Shavrukov 1993)
- ▶ Lyndon (Shamkanov, 2011)

Definition

For a sequent Γ_1, Γ_2 , the expression of the form $\Gamma_1 \mid \Gamma_2$ is called its splitting.

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An interpolant of a split sequent $\Gamma_1 \mid \Gamma_2$ is defined as an interpolant of the formula $\overline{\Gamma}_1^\# \rightarrow \Gamma_2^\#$.

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Note that an interpolant of a split sequent $\overline{A} \mid B$ is an interpolant for the formula $A \rightarrow B$.

Interpolation: proof-theoretic strategy

Basic observations

- ▶ Given an application of an inference rule, every splitting of the conclusion produces splittings of the premises preserving ancestor relationship. For example:

$$\square \frac{\Gamma_1, \Diamond \Gamma_1, A \mid \Gamma_2, \Diamond \Gamma_2}{\Diamond \Gamma_1, \square A, \Delta_1 \mid \Diamond \Gamma_2, \Delta_2} ;$$

- ▶ Moreover, there is an explicit definition of the interpolant for the split sequent in the conclusion from interpolants of the split sequents in the premises.

Interpolation: proof-theoretic strategy

- ▶ Given a proof of a sequent Γ_1, Γ_2 , construct an interpolant for the split sequent $\Gamma_1 \mid \Gamma_2$ by induction on the structure of the proof.

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But circular proofs are not well-founded!

Proof sketch:

- ▶ given a circular proof Γ_1, Γ_2 , construct a split circular proof of $\Gamma_1 \mid \Gamma_2$
- ▶ from the split circular proof, construct an interpolant C by the fixed-point theorem
- ▶ prove that C is indeed an interpolant using admissibility of the Löb's rule and that it satisfies restrictions on occurrences of propositional variables

Proof sketch:

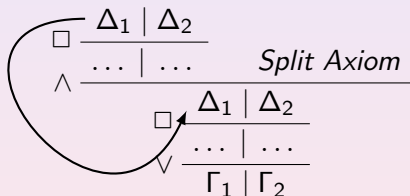
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Proposition

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Proof sketch:

- ▶ from the split circular proof, construct an interpolant C by the fixed-point theorem

$$(\top) \Gamma_1 \mid \top, \Gamma_2 \quad (\perp) \Gamma_1, \top \mid \Gamma_2$$

$$(\perp) \Gamma_1, A, \bar{A} \mid \Gamma_2 \quad (\bar{A}) \Gamma_1, A \mid \bar{A}, \Gamma_2 \quad (\top) \Gamma_1 \mid A, \bar{A}, \Gamma_2$$

$$\wedge_l \frac{(C) \Gamma_1, A \mid \Gamma_2 \quad (D) \Gamma_1, B \mid \Gamma_2}{(C \vee D) \Gamma_1, A \wedge B \mid \Gamma_2} \quad \vee_l \frac{(C) \Gamma_1, A, B \mid \Gamma_2}{(C) \Gamma_1, A \vee B \mid \Gamma_2}$$

$$\wedge_r \frac{(C) \Gamma_1 \mid \Gamma_2, A \quad (D) \Gamma_1 \mid \Gamma_2, B}{(C \wedge D) \Gamma_1 \mid \Gamma_2, A \wedge B} \quad \vee_r \frac{(C) \Gamma_1 \mid \Gamma_2, A, B}{(C) \Gamma_1 \mid \Gamma_2, A \vee B}$$

$$\Box_l \frac{(C) \Gamma_1, \Diamond \Gamma_1, A \mid \Gamma_2, \Diamond \Gamma_2}{(\Diamond C) \Diamond \Gamma_1, \Box A, \Delta_1 \mid \Diamond \Gamma_2, \Delta_2}$$

$$\Box_r \frac{(C) \Gamma_1, \Diamond \Gamma_1 \mid \Gamma_2, \Diamond \Gamma_2, A}{(\Box C) \Diamond \Gamma_1, \Delta_1 \mid \Diamond \Gamma_2, \Box A, \Delta_2}$$

From split proofs to systems of equations

The diagram illustrates the Split Axiom as a system of equations. It features three main components arranged in a cycle:

- Top Component:** A fraction with $\Delta_1 \mid \Delta_2$ in the numerator and $\Lambda_1 \mid \Lambda_2$ in the denominator. A left curly brace $\{$ is positioned to the left of the fraction.
- Middle Component:** A fraction with $\dots \mid \dots$ in the numerator and $\Lambda_1 \mid \Lambda_2$ in the denominator. A left square bracket $[$ is positioned to the left of the fraction.
- Bottom Component:** A fraction with $\Delta_1 \mid \Delta_2$ in the numerator, $\dots \mid \dots$ in the middle, and $\Gamma_1 \mid \Gamma_2$ in the denominator. A left square bracket $[$ is positioned to the left of the fraction.

A horizontal line separates the middle and bottom components, with the text *Split Axiom* to its right. Arrows indicate dependencies: a curved arrow from the top component to the middle component, a curved arrow from the bottom component to the middle component, and a curved arrow from the bottom component back to the top component.

From split proofs to systems of equations

$$\begin{array}{c}
 \wedge \frac{(X_1) \Delta_1 \mid \Delta_2 \quad (X_2) \Lambda_1 \mid \Lambda_2}{\dots \mid \dots} \\
 \square \frac{\dots \mid \dots}{\Lambda_1 \mid \Lambda_2} \quad \wedge \frac{\Delta_1 \mid \Delta_2}{\dots \mid \dots} \\
 \vee \frac{\dots \mid \dots}{\Gamma_1 \mid \Gamma_2}
 \end{array}
 \quad \text{Split Axiom}$$

The diagram illustrates the Split Axiom. It features three main components arranged vertically. The top component is a conjunction \wedge over a horizontal line, with two sub-proofs: $(X_1) \Delta_1 \mid \Delta_2$ and $(X_2) \Lambda_1 \mid \Lambda_2$. Below this is a boxed disjunction \square over a horizontal line, with $\dots \mid \dots$ above and $\Lambda_1 \mid \Lambda_2$ below. The bottom component is a disjunction \vee over a horizontal line, with $\dots \mid \dots$ above and $\Gamma_1 \mid \Gamma_2$ below. A curved arrow points from the top conjunction to the boxed disjunction. Another curved arrow points from the boxed disjunction to the bottom disjunction. A third curved arrow points from the bottom disjunction back to the top conjunction, completing a cycle.

From split proofs to systems of equations

$$\begin{array}{c}
 \wedge \frac{(X_1) \Delta_1 \mid \Delta_2 \quad (X_2) \Lambda_1 \mid \Lambda_2}{\dots \mid \dots} \\
 \square \frac{\quad}{(A_2(X_1, X_2)) \Lambda_1 \mid \Lambda_2} \quad \text{Split Axiom} \\
 \wedge \frac{\quad}{(A_1(X_1, X_2)) \Delta_1 \mid \Delta_2} \\
 \vee \frac{\dots \mid \dots}{\Gamma_1 \mid \Gamma_2}
 \end{array}$$

The diagram illustrates a system of equations for split proofs. It shows a sequence of logical operations: a conjunction (\wedge) at the top, followed by a box (\square) representing a split, then another conjunction (\wedge), and finally a disjunction (\vee) at the bottom. The equations are connected by arrows, indicating a flow or dependency between the different parts of the proof system.

From split proofs to systems of equations

$$\begin{array}{c}
 \wedge \frac{(X_1) \Delta_1 \mid \Delta_2 \quad (X_2) \Lambda_1 \mid \Lambda_2}{\dots \mid \dots} \quad X_2 \leftrightarrow A_2(X_1, X_2) \\
 \square \frac{\quad}{(A_2(X_1, X_2)) \Lambda_1 \mid \Lambda_2} \quad \textit{Split Axiom} \\
 \wedge \frac{\quad}{(A_1(X_1, X_2)) \Delta_1 \mid \Delta_2} \\
 X_1 \leftrightarrow A_1(X_1, X_2) \quad \vee \frac{\dots \mid \dots}{\Gamma_1 \mid \Gamma_2}
 \end{array}$$

Fixed-Point Theorem (de Jongh, 1975, Sambin, 1975)

Let $A(P)$ be a formula in which P only occurs within the scope of modality. Then there is a formula F such that $Var(F) \subset Var(A) \setminus \{P\}$ and

$$GL \vdash \Box(P \leftrightarrow A(P)) \leftrightarrow \Box(P \leftrightarrow F) .$$

Moreover, if $A(P)$ is positive in P , then $Var^+(F) \subset Var(A)^+ \setminus \{P\}$ and $Var^-(F) \subset Var(A)^-$.

$\Box B$ is an abbreviation for $B \wedge \Box B$.