Circular Proofs for Gödel-Löb Logic

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June 7, 2013

Gödel-Löb logic

Axioms

- Boolean tautologies
- $\Box(A \to B) \to (\Box A \to \Box B)$
- $\Box(\Box A \to A) \to \Box A$ (Löb's axiom)

Rules

$$\frac{A, A \to B}{B} \qquad \frac{A}{\Box A}$$

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GL is sound and complete w.r.t. the arithmetical semantics, where $\Box A$ corresponds to "A is provable in Peano arithmetic".

Sequent calculus syntax

Formulas are are given by:

$$A ::= P \mid \overline{P} \mid \top \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid \Box A \mid \Diamond A \,.$$

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Sequents are finite multisets of formulas.

For a sequent $\Gamma = A_1, \ldots, A_n$, its intended interpretation as a formulas is

$$\Gamma^\sharp := \begin{cases} \bot & \text{if } n = 0, \\ A_1 \vee \ldots \vee A_n & \text{otherwise.} \end{cases}$$

Standard sequent calculus $K4_{Seq}$

Axioms

$$\Gamma, A, \overline{A} \qquad \Gamma, \top$$

Rules

$$\wedge\,\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\Box\,\frac{\Gamma, \Diamond\Gamma, A}{\Diamond\Gamma, \Box A, \Delta}$$

Standard sequent calculus $\mathrm{GL_{Seq}}$

$GL_{Seq} = K4_{Seq}$ + the modal rule $\Box_{GL}$

$$\Box_{GL} \; \frac{\Gamma, \Diamond\Gamma, \Diamond\overline{A}, A}{\Diamond\Gamma, \Box A, \Delta}$$

Circular sequent calculus $\mathsf{GL}_{\mathsf{CSeq}}$

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Circular sequent calculus $\mathrm{GL}_{\mathrm{CSeq}}$

$GL_{CSeq} = K4_{Seq} +$ circular proofs

A circular proof of the Löb's axiom

$$\vee\,\dfrac{\Box\,\dfrac{\wedge\,\dfrac{\Box\,\dfrac{\Box P \wedge \overline{P}, \Diamond(\Box P \wedge \overline{P}), P}{\Box P, \Diamond(\Box P \wedge \overline{P}), P} \qquad \overline{P}, \Diamond(\Box P \wedge \overline{P}), P}{\Box P \wedge \overline{P}, \Diamond(\Box P \wedge \overline{P}), P}}{\Diamond(\Box P \wedge \overline{P}), \Box P}}{\Diamond(\Box P \wedge \overline{P}) \vee \Box P}$$

Theorem

$GL_{Seq} \vdash \Gamma \Leftrightarrow GL_{CSeq} \vdash \Gamma$

An application: Lyndon interpolation syntactically

Lyndon interpolation

Craig interpolation

If $L \vdash A \to B$, then there is a formula C containing only common variables of A and B such that $L \vdash A \to C$, $L \vdash C \to B$.

Lyndon interpolation

is a strengthening of the Craig one by the additional requirement:

every propositional variable that has a positive (negative) occurrence in C must also have positive (negative) occurrences both in A and B.

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Interpolation properties of GL

- Craig (Smoryński 1978, Boolos 1979)

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Interpolation properties of *GL*

- Craig (Smoryński 1978, Boolos 1979)
- Uniform (Shavrukov 1993)
- Lyndon (Shamkanov, 2011)

Definition

For a sequent Γ_1, Γ_2, the expression of the form $\Gamma_1 \mid \Gamma_2$ is called its splitting.

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An interpolant of a split sequent $\Gamma_1 \mid \Gamma_2$ is defined as an interpolant of the formula $\overline{\Gamma}_1^\sharp \to \Gamma_2^\sharp$.

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Note that an interpolant of a split sequent $\overline{A} \mid B$ is an interpolant for the formula $A \to B$.

Interpolation: proof-theoretic strategy

Basic observations

- Given an application of an inference rule, every splitting of the conclusion produces splittings of the premises preserving ancestor relationship. For example:

$$\Box \ \frac{\Gamma_1, \Diamond\Gamma_1, A \mid \Gamma_2, \Diamond\Gamma_2}{\Diamond\Gamma_1, \Box A, \Delta_1 \mid \Diamond\Gamma_2, \Delta_2} \ ;$$

- Moreover, there is an explicit definition of the interpolant for the split sequent in the conclusion from interpolants of the split sequents in the premises.

Interpolation: proof-theoretic strategy

- Given a proof of a sequent Γ_1, Γ_2, construct an interpolant for the split sequent $\Gamma_1 \mid \Gamma_2$ by induction on the structure of the proof.

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  But circular proofs are not well-founded!

Proof sketch:

- given a circular proof Γ_1, Γ_2, construct a split circular proof of $\Gamma_1 \mid \Gamma_2$
- from the split circular proof, construct an interpolant C by the fixed-point theorem
- prove that C is indeed an interpolant using admissibility of the Löb's rule and that it satisfies restrictions on occurrences of propositional variables

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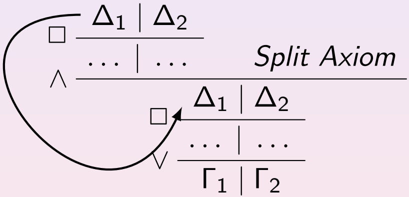


Proof sketch:

- from the split circular proof, construct an interpolant C by the fixed-point theorem

$$(\top)\ \ \Gamma_1 \mid \top, \Gamma_2 \qquad\qquad (\bot)\ \ \Gamma_1, \top \mid \Gamma_2$$

$$(\bot)\ \ \Gamma_1, A, \overline{A} \mid \Gamma_2 \qquad\qquad (\overline{A})\ \ \Gamma_1, A \mid \overline{A}, \Gamma_2 \qquad\qquad (\top)\ \ \Gamma_1 \mid A, \overline{A}, \Gamma_2$$

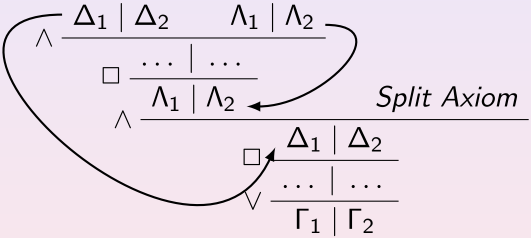
$$\wedge_l\ \frac{(C)\ \ \Gamma_1, A \mid \Gamma_2 \qquad (D)\ \ \Gamma_1, B \mid \Gamma_2}{(C \vee D)\ \ \Gamma_1, A \wedge B \mid \Gamma_2} \qquad \vee_l\ \frac{(C)\ \ \Gamma_1, A, B \mid \Gamma_2}{(C)\ \ \Gamma_1, A \vee B \mid \Gamma_2}$$

$$\wedge_r\ \frac{(C)\ \ \Gamma_1 \mid \Gamma_2, A \qquad (D)\ \ \Gamma_1 \mid \Gamma_2, B}{(C \wedge D)\ \ \Gamma_1 \mid \Gamma_2, A \wedge B} \qquad \vee_r\ \frac{(C)\ \ \Gamma_1 \mid \Gamma_2, A, B}{(C)\ \ \Gamma_1 \mid \Gamma_2, A \vee B}$$

$$\Box_l\ \frac{(C)\ \ \Gamma_1, \Diamond\Gamma_1, A \mid \Gamma_2, \Diamond\Gamma_2}{(\Diamond C)\ \ \Diamond\Gamma_1, \Box A, \Delta_1 \mid \Diamond\Gamma_2, \Delta_2}$$

$$\Box_r\ \frac{(C)\ \ \Gamma_1, \Diamond\Gamma_1 \mid \Gamma_2, \Diamond\Gamma_2, A}{(\Box C)\ \ \Diamond\Gamma_1, \Delta_1 \mid \Diamond\Gamma_2, \Box A, \Delta_2}$$

From split proofs to systems of equations



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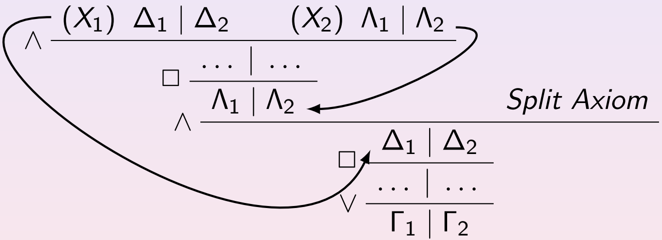

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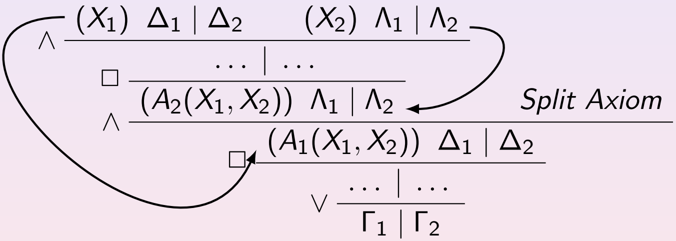

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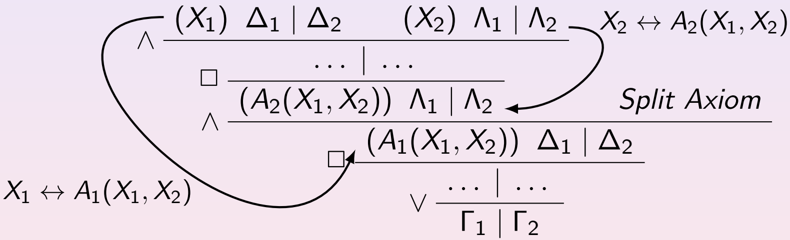

Fixed-Point Theorem (de Jongh, 1975, Sambin, 1975)

Let A(P) be a formula in which P only occurs within the scope of modality. Then there is a formula F such that $Var(F) \subset Var(A) \setminus \{P\}$ and

$$GL \vdash \boxdot(P \leftrightarrow A(P)) \leftrightarrow \boxdot(P \leftrightarrow F) \, .$$

Moreover, if $A(P)$ is positive in P, then $Var^{+}(F) \subset Var(A)^{+} \setminus \{P\}$ and $Var^{-}(F) \subset Var(A)^{-}$.

$\boxdot B$ is an abbreviation for $B \wedge \Box B$.