Hamming metrics and products of modal logics

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Joint work with Andrey Kudinov and Valentin Shehtman.

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$ML(\Diamond)$

Modal formulas (the unimodal case):

$PV = \{p_1, p_2, \dots\}$ — variables;

$\neg, \vee$;

$\Diamond$ — unary connective.

($\Box\varphi$ — abbreviation for $\neg\Diamond\neg\varphi$.)

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$\neg, \vee$;
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($\Box\varphi$ — abbreviation for $\neg\Diamond\neg\varphi$.)

frame: $\mathsf{F} = (W, R)$, where $W \neq \varnothing$, $R \subseteq W \times W$.
valuation: $\theta : PV \to \mathcal{P}(W)$.
model: (F, θ).

$$\begin{array}{lcl} \mathsf{M}, w \vDash p & \Longleftrightarrow & w \in \theta(p); \\ \mathsf{M}, w \vDash \neg\varphi & \Longleftrightarrow & \mathsf{M}, w \nvDash \varphi; \\ \mathsf{M}, w \vDash \varphi \vee \psi & \Longleftrightarrow & \mathsf{M}, w \vDash \varphi \text{ or } \mathsf{M}, w \vDash \psi; \\ \mathsf{M}, w \vDash \Diamond\varphi & \Longleftrightarrow & \exists v (wRv \;\&\; \mathsf{M}, v \vDash \varphi). \end{array}$$

$ML(\Diamond_0, \dots, \Diamond_{n-1})$

n-modal formulas:
$PV = \{p_1, p_2, \dots\}$ — variables;
$\neg, \vee$;
$\Diamond_0, \dots, \Diamond_{n-1}$ — unary connectives.

n-frame: $(W, R_0, \dots, R_{n-1})$.

$$\mathrm{M}, w \vDash \Diamond_i \varphi \qquad \Longleftrightarrow \qquad \exists v (w R_i v \;\&\; \mathrm{M}, v \vDash \varphi).$$

A – is an alphabet (a nonempty set),
$\mathbf{x} = (x_0, \ldots, x_{n-1}), \mathbf{y} = (y_0, \ldots, y_{n-1}) \in A^n$

$h(\mathbf{x}, \mathbf{y})$ is the *Hamming distance* between $\mathbf{x}$ and $\mathbf{y}$:

$$h(\mathbf{x}, \mathbf{y}) = |\{i \mid x_i \neq y_i\}|.$$

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Consider frames (A^n, H), $(A^n, \overline{H})$:
for $\mathbf{x}, \mathbf{y} \in A^n$,

$$\mathbf{x} H \mathbf{y} \iff h(\mathbf{x}, \mathbf{y}) = 1.$$

$$\mathbf{x} \overline{H} \mathbf{y} \iff h(\mathbf{x}, \mathbf{y}) \leq 1.$$

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$H(x) = \{y \mid xHy\}$ is the sphere with the center x of radius 1.
$\overline{H}(x) = \{y \mid x\overline{H}y\}$ is the ball with the center x of radius 1.

Balls and spheres in Hamming spaces.

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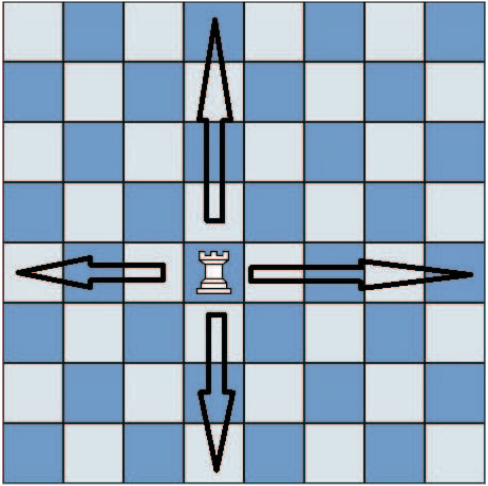
$|A| = 8$, $n = 2$, a ball of radius 1:



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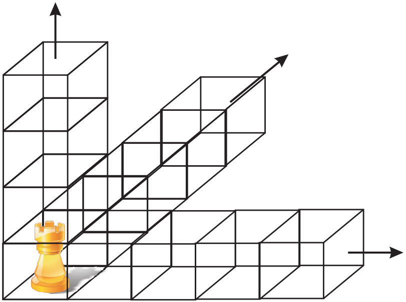

The ball with the center x of radius r is the set of all points that the (space)rook can reach from x in at most r moves.

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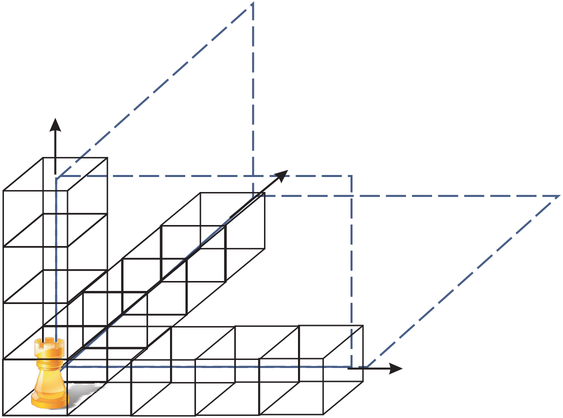

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Modal products.

$(W_1, R_1) \times \cdots \times (W_n, R_n)$ is the n-frame

$$(W_1 \times \cdots \times W_n, R'_1, \ldots, R'_n),$$

where
$(w_1, \ldots, w_n) R'_i (v_1, \ldots, v_n) \iff w_i R_i v_i$ and $w_k = v_k$ for $k \neq i$.

For logics $L_1, \ldots, L_n$,
$L_1 \times \cdots \times L_n := Log(\{F_1 \times \cdots \times F_n \mid F_1 \vDash L_1, \ldots, F_n \vDash L_n\})$.

Consider a frame $(A, \neq)$ and the product

$$(A, \neq)^n = (A^n, \neq_0, \ldots, \neq_{n-1}),$$

i.e.,

$$\mathbf{x} \neq_i \mathbf{y} \iff x_i \neq y_i \text{ and } x_j = y_j \text{ for } j \neq i.$$

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$$(A^n, H) = (A^n, \neq_0 \cup \cdots \cup \neq_{n-1}).$$

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$t^{(n)} : ML(\Diamond) \to ML(\Diamond_0, \dots, \Diamond_{n-1})$

$$t^{(n)}(\Diamond\varphi) := \Diamond_0 t^{(n)}(\varphi) \vee \dots \vee \Diamond_{n-1} t^{(n)}(\varphi).$$

Trivial fact: if $A \neq \varnothing$, $n > 0$, $\theta : PV \to \mathcal{P}(A^n)$, φ, $\mathbf{x} \in A^n$, then

$$((A, \neq)^n, \theta), \mathbf{x} \vDash t^{(n)}(\varphi) \iff ((A^n, H), \theta), \mathbf{x} \vDash \varphi;$$

so $Log(A^n, H)$ is a fragment of $Log((A, \neq)^n)$.

Consider a frame $(A, A \times A)$ and the product

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$$\overline{t}^{(n)}(\Diamond\varphi) := \Diamond_0 \overline{t}^{(n)}(\varphi) \vee \cdots \vee \Diamond_{n-1} \overline{t}^{(n)}(\varphi).$$

$$((A, A \times A)^n, \theta), \mathbf{x} \models \overline{t}^{(n)}(\varphi) \iff ((A^n, \overline{H}), \theta), \mathbf{x} \models \varphi.$$

$Log(A^n, \overline{H})$ is a fragment of $Log((A, A \times A)^n)$.

- Non-finite axiomatizability
- Undecidability
- Decidability

Non-finite axiomatizability

Definition

Let L be a modal logic. For $m \geq 1$, put

$$L\lceil m = \{\varphi \in L \mid PV(\varphi) \subseteq \{p_1, \ldots, p_m\}\}, \quad L\lceil 0 = \{\varphi \in L \mid PV(\varphi) = \varnothing\}.$$

These sets of formulas are called *the m-fragments of L*.

Theorem

If A is infinite, then for any $n > 0$ the logic $Log(A^n, H)$ is not axiomatizable by any of its m-fragments.

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Remark:

This gives us a simple example of a modal logic which is not finitely axiomatizable, but has a finitely axiomatizable conservative extension: the logic $Log(\mathbb{R}^2, \neq)$ is not f.a., whereas some topological modal logics with the difference modality of $\mathbb{R}^2$ have finite axiomatizations [Kudinov, 2005], [Kudinov, Shehtman, 2011].

Corollary

Suppose B is nonempty, A is infinite. Then $Log((A, \neq) \times (B, \neq))$ is not finite-variable axiomatizable.

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Recently [C.Hampson, A. Kurucz, 2012] a similar result was obtained for products with the minimal difference logic $\mathbf{DL}$: all logics in the interval between $\mathbf{K} \times \mathbf{DL}$ and $\mathbf{S5} \times \mathbf{DL}$ are not finite-variable axiomatizable.

Undecidability

Theorem

For $n \geq 2$, there exists a polynomial-time translation $f^{(n)}$ such that for any n-modal formula φ we have:

$$\varphi \ \ is \ \ (A, A \times A)^n\text{-satisfiable} \iff f^{(n)}(\varphi) \text{ is } (A^n, H)\text{-satisfiable.}$$

Corollary

Let $\mathfrak{A}$ be a class of nonempty sets, $n > 0$. If the logic $Log(\{(A, A \times A)^n \mid A \in \mathfrak{A}\})$ is undecidable, then the logic $Log(\{(A^n, H) \mid A \in \mathfrak{A}\})$ is undecidable.

Since all logics $\mathbf{S5}^n$, $n > 2$, are undecidable, we have

Corollary

If a class of nonempty sets $\mathfrak{A}$ contains an infinite set, then the logic $Log(\{(A^n, H) \mid A \in \mathfrak{A}\})$ is undecidable.

Decidability

TB is the logic of all (finite) symmetric reflexive frames.

DB is the logic of all (finite) symmetric serial (i.e., $\forall x \exists y \; xRy$ holds) frames.

These logics have finite axiomatizations (very simple).
So they are decidable (since they have the fmp); moreover — are PSPACE-complete).

Decidability

For functions $f, g : I \to A$, put

$$f\,H\,g \iff |\{i \mid i \in I,\ f(i) \neq g(i)\}| = 1.$$

Theorem

$Log(\{0,1\}^\omega, H) = \boldsymbol{DB}$.

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$$U \triangle_1 V \iff U \triangle V \text{ is a singleton.}$$

Corollary

$Log(\mathcal{P}(\omega), \triangle_1) = \boldsymbol{DB}$.

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$$f \, \overline{H} \, g \iff f \, H \, g \text{ or } f = g.$$

$$S \triangle_{\leq 1} S' \iff |S \triangle S'| \leq 1.$$

Corollary

If I is infinite, $|A| > 1$, then

$$Log(A^I, \overline{H}) = \boldsymbol{TB}; \quad Log(\mathcal{P}(I), \triangle_{\leq 1}) = \boldsymbol{TB}.$$

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Some open problems

Problem

Does there exist a finitely axiomatizable logic $Log(\omega^n, \overline{H})$*, for* $n = 2, 3, \ldots$*?*

$Log(\omega^2, \overline{H})$ is decidable, because $\mathbf{S5}^2$ is decidable; $Log(\omega^2, H)$ is decidable, because $Log((\omega, \neq)^2)$ is decidable. For $n > 2$, $Log((\omega, \neq)^2)$ is undecidable, since $\mathbf{S5}^n$ is undecidable.

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$$\boldsymbol{TB} = Log(2^\omega, \overline{H}) \stackrel{?}{=} \bigcap_{n<\omega} Log(2^n, \overline{H})$$

Non-finite axiomatizability

Let L be a logic.

$$L\lceil m = \{\varphi \in L \mid PV(\varphi) \subseteq \{p_1, \ldots, p_m\}\}.$$

Frames F, G are called *m-equivalent* (notation: $\mathrm{F} \sim_m \mathrm{G}$), if $Log(\mathrm{F})\lceil m = Log(\mathrm{G})\lceil m$.

Proposition (L. Maksimova, D. Skvortsov, V. Shehtman, 1979)

Consider a logic Λ and suppose that for every m there exist frames G_m, G'_m such that $G_m \sim_m G'_m$, $\Lambda \subseteq Log(G_m)$, $\Lambda \not\subseteq Log(G'_m)$. Then Λ is not finitely axiomatizable, and moreover, not axiomatizable by any of its m-fragment.

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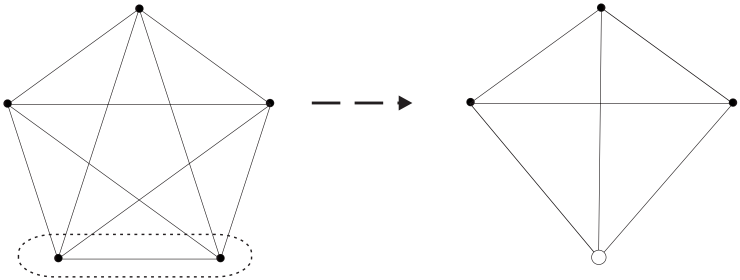
$\mathsf{K}_m := (\{0, \dots, m-1\}, \neq)$,
$\mathsf{K}'_m := (\{0, \dots, m-1\}, R_m)$, where $iR_mj \iff i \neq j$ or $i = j = m-1$.
For any $m \geq 0$ we have: $\mathsf{K}_{2^m+1} \sim_m \mathsf{K}'_{2^m}$.
For an infinite A, for any $m \geq 0$:
$Log(A^n, H) \not\subseteq \mathsf{K}_m$,
$Log(A^n, H) \subseteq Log(\mathsf{K}'_m)$.



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We will “restore” the modalities of $(A, A \times A)^n$ via the unimodal box interpreted by the H relation.

Undecidability

$\Box^0\varphi = \varphi$, $\Box^{l+1}\varphi = \Box\Box^l\varphi$, $\Box^{\leq l}\varphi = \bigwedge_{0\leq i\leq l} \Box^i\varphi$.

$\Box^{\leq n}$ acts like the universal modality on (A^n, H).

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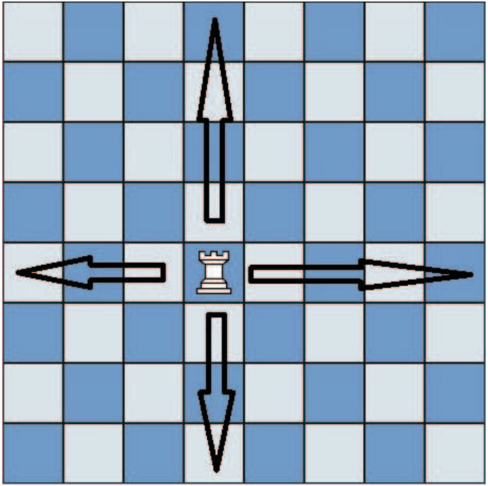

For each set $U \subseteq n = \{0, \dots, n-1\}$ we fix a variable p_U.
Let $sets^{(n)}$ be the conjunction of the following formulas:

$$p_\varnothing \wedge \neg\Diamond p_\varnothing \tag{1}$$

$$\Box^{\leq n}\left(\bigvee_{U\subseteq n} p_U \wedge \bigwedge_{U,V\subseteq n,\ U\neq V}(p_U \to \neg p_V)\right) \tag{2}$$

$$\Box^{\leq n}\left(\bigwedge_{U,V\subseteq n,\ |U\triangle V|=1}(p_U \to \Diamond p_V)\right) \tag{3}$$

$$\Box^{\leq n}\left(\bigwedge_{U,V\subseteq n,\ |U\triangle V|>1}(p_U \to \neg\Diamond p_V)\right) \tag{4}$$

(Note that if we also add the conjuncts $p_U \to \neg\Diamond p_U$ for all nonempty $U \subseteq n$, then we obtain the frame formula for the frame $(\mathcal{P}(n), \triangle_1)$ at the point $\varnothing$.)

$$sets^{(n)} := p_\varnothing \wedge \neg\Diamond p_\varnothing \wedge \Box^{\leq n}\left(\bigvee_{U\subseteq n} p_U \wedge \bigwedge_{U,V\subseteq n,\ U\neq V}(p_U \to \neg p_V)\right) \wedge$$

$$\Box^{\leq n}\left(\bigwedge_{U,V\subseteq n,\ |U\triangle V|=1}(p_U \to \Diamond p_V) \wedge \bigwedge_{U,V\subseteq n,\ |U\triangle V|>1}(p_U \to \neg\Diamond p_V)\right)$$

The meaning of the formula $sets^{(n)}$ is explained by the following key fact.

Lemma

Let $|A| > 1$, $((A^n, H), \theta), \mathbf{r} \models sets^{(n)}$. Then there exists a unique permutation $\sigma : n \to n$ such that for any $\mathbf{x} \in A^n$ and $V \subseteq n$,

$$((A^n, H), \theta), \mathbf{x} \models p_V \iff D_\sigma(\mathbf{r}, \mathbf{x}) = V,$$

where

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$$plane_i^{(n)} := \bigvee_{U \subseteq n,\ i \notin U} p_U.$$

$$\begin{aligned}\Diamond_i^{(n)}\varphi \quad = \quad & \varphi \vee ((plane_i^{(n)} \to \Diamond(\neg plane_i^{(n)} \wedge \varphi))\wedge \\ & \wedge(\neg plane_i^{(n)} \to \Diamond(plane_i^{(n)} \wedge (\varphi \vee (\Diamond(\neg plane_i^{(n)} \wedge \varphi)))))).\end{aligned}$$

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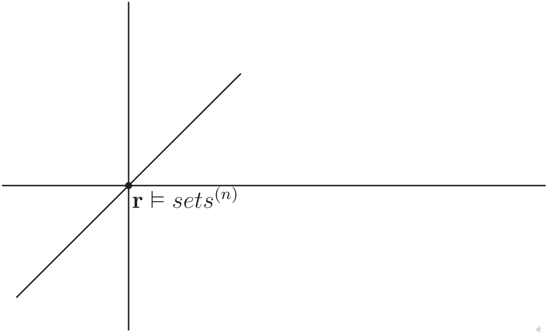

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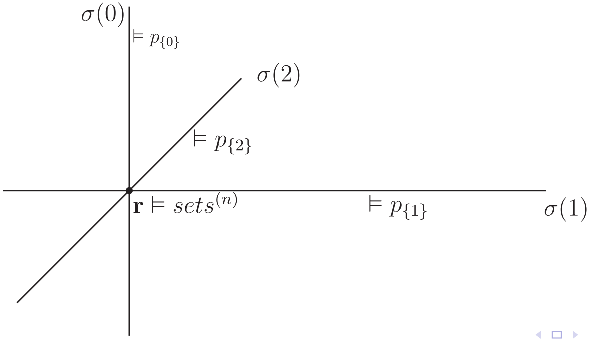

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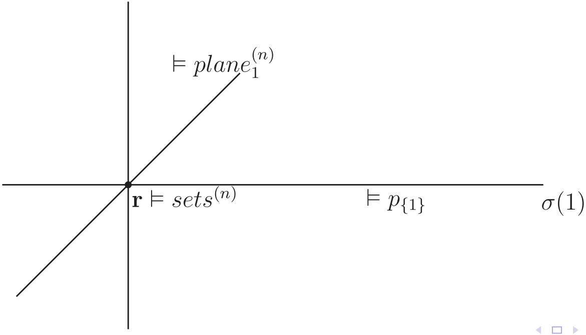

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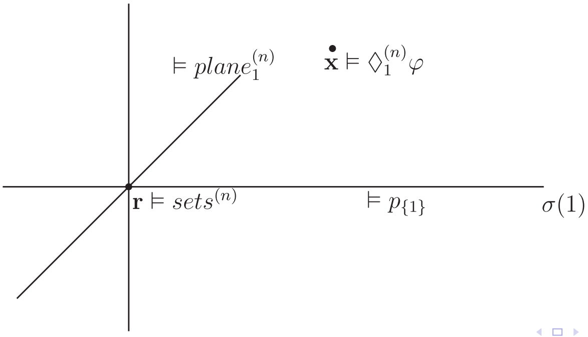

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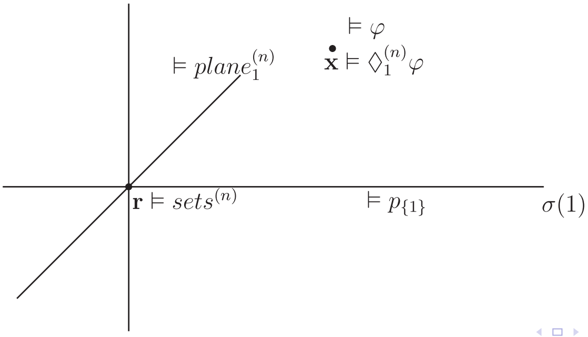

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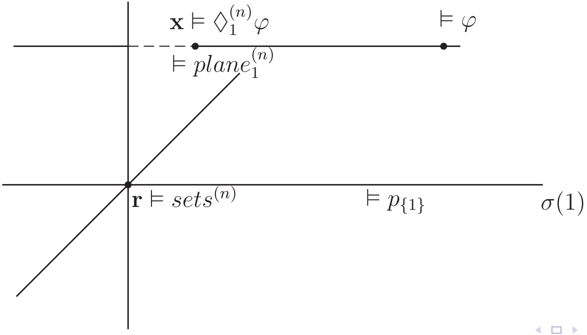
$$\Diamond_i^{(n)}\varphi = \varphi \vee ((plane_i^{(n)} \to \Diamond(\neg plane_i^{(n)} \wedge \varphi)) \wedge \\ \wedge(\neg plane_i^{(n)} \to \Diamond(plane_i^{(n)} \wedge (\varphi \vee (\Diamond(\neg plane_i^{(n)} \wedge \varphi)))))).$$



$D_\sigma(\mathbf{x},\mathbf{y}) := \{i \mid x_{\sigma(i)} \neq y_{\sigma(i)}\};$
$D_\sigma(\mathbf{r},\mathbf{x}) = V \iff ((A^n, H), \theta), \mathbf{x} \vDash p_V.$

$$plane_i^{(n)} := \bigvee_{U \subseteq n,\ i \notin U} p_U.$$

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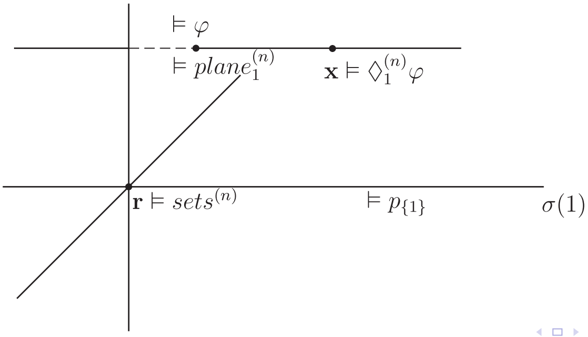
$$\wedge(\neg plane_i^{(n)} \to \Diamond(plane_i^{(n)} \wedge (\varphi \vee (\Diamond(\neg plane_i^{(n)} \wedge \varphi)))))).$$



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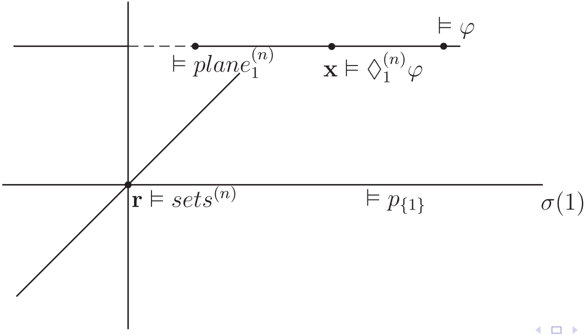

For a formula φ in the n-modal language $ML(\Diamond_0, \ldots, \Diamond_{n-1})$, we define the unimodal formula $[\varphi]^{(n)}$:

$$[p]^{(n)} = p \text{ for } p \in PV; \quad [\phi \wedge \psi]^{(n)} = [\phi]^{(n)} \wedge [\psi]^{(n)}; \quad [\neg\phi]^{(n)} = \neg([\phi]^{(n)});$$
$$[\Diamond_i \phi]^{(n)} = \Diamond_i^{(n)} [\phi]^{(n)}.$$

If $|A| > 1$, $((A^n, H), \theta), \mathbf{r} \vDash sets^{(n)}$, $(A, A \times A)^n = (A^n, R_0, \ldots R_{n-1})$, then for any n-modal formula φ with $PV(\varphi) \cap PV(sets^{(n)}) = \varnothing$, for any $\mathbf{x} \in A^n$, we have

$$((A^n, R_{\sigma(0)}, \ldots R_{\sigma(n-1)}), \theta), \mathbf{x} \vDash \varphi \iff ((A^n, H), \theta), \mathbf{x} \vDash [\varphi]^{(n)}$$

Theorem

For $|A| > 1$, $n \geq 2$, for any n-modal formula φ that does not share variables with $sets^{(n)}$, we have:

$$\varphi \;\; is \;\; (A, A \times A)^n\text{-satisfiable} \iff sets^{(n)} \wedge [\varphi]^{(n)} \text{ is } (A^n, H)\text{-satisfiable.}$$

Some open problems

Problem

Does there exist a finitely axiomatizable logic $Log(\omega^n, \overline{H})$*, for* $n = 2, 3, \ldots$ *?*

$Log(\omega^2, \overline{H})$ is decidable, because $\mathbf{S5}^2$ is decidable; $Log(\omega^2, H)$ is decidable, because $Log((\omega, \neq)^2)$ is decidable. For $n > 2$, $Log((\omega, \neq)^2)$ is undecidable, since $\mathbf{S5}^n$ is undecidable.

Problem

Is there a decidable logic $Log(\omega^n, \overline{H})$*,* $n = 3, 4, \ldots$ *?*

Problem

$$\boldsymbol{TB} = Log(2^\omega, \overline{H}) \stackrel{?}{=} \bigcap_{n<\omega} Log(2^n, \overline{H})$$

Thank you!